

# Bol-loops of order $3 \cdot 2^n$

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## Abstract

In this article we construct proper Bol-loops of order  $3 \cdot 2^n$  using a generalisation of the semidirect product of groups defined by Birkenmeier and Xiao. Moreover we classify the obtained loops up to isomorphism.

**Keywords:** Bol-loop, loop, group, semidirect product.

**Mathematics Subject Classification:** 20N05.

## Introduction

Burn proofs in [3] that the smallest proper Bol-loops are of order 8. But they can not be constructed as a semidirect product defined in [1]. The smallest proper Bol-loops which can be constructed using a semidirect product as defined in this article have order 12. Up to isomorphism these loops can be realised as semidirect product of the cyclic group of order 3 and the elementary abelian groups of order 4. There are no proper Bol-loops of order 9, 10 or 11. It seems that order 12 plays an interesting role in the theory of loops since the smallest proper Moufang-loop has also order 12 (cf. [5]).

A *loop* is a set  $L$  with a binary operation  $\cdot$ , a neutral element 1 and unique solutions of the equations  $x \cdot a = b$  and  $a \cdot x = b$ . The loop  $L$  is a *left Bol-loop* if  $((x \cdot y)z)y = x((y \cdot z)y)$  for all  $x, y, z \in L$  holds. Analogously one defines a right Bol-loop by the identity  $x(y(x \cdot z)) = (x(y \cdot x))z$ .

In this paper we consider a special case of the semidirect product of loops defined by Birkenmeier and Xiao in [2]. Starting with groups  $N$  and  $Q$  we obtain a loop  $L$  on  $N \rtimes Q = \{(a, p) : a \in N, p \in Q\}$ . The multiplication  $*$  of  $L$  is defined as  $(a, p) * (b, q) = (a^{\Phi(q)} \circ b^{\Psi(p)}, p \bullet q)$ , where  $\circ$  and  $\bullet$  are the multiplications of  $N$  and  $Q$ . The mapping  $\Phi(p)$  respectively  $\Psi(p)$  from  $N$  into  $N$  is determined by a mapping  $\Phi$  respectively  $\Psi$  from  $Q$  into the set of mappings from  $N$  into  $N$ . According to [2] we know that  $(L, *)$  is a loop with neutral element  $(1, 1)$  if  $\Phi(p)$  and  $\Psi(p)$  are bijective,  $1^{\Phi(p)} = 1^{\Psi(p)} = 1$  holds for all  $p \in Q$  and  $\Phi(1) = \Psi(1) = \text{id}_N$ . The constructed loops are associative if and only if the mappings  $\Psi$ ,  $\Phi$ ,  $\Phi(p)$  and  $\Psi(p)$  are homomorphisms and  $\Phi(p)$  and  $\Psi(q)$  commute for all  $p, q \in Q$ .

Although the semidirect product treated by us here is a special case of the semidirect products defined in [1], [2] and [9] the construction presented here yields in general loops with no further identities. For example the 15 non-associative loops  $L = C_3 \rtimes C_3$  of order 9, which are the smallest possible examples, are not even power associative and only three of them are commutative.

## Bol-loops of order $3 \cdot 2^n$

We now construct loops of the form  $L = C_3 \rtimes (C_2)^n$ . These loops are all power-associative and under certain conditions Bol-loops.

**Remark 1.** The only two mappings of  $C_3$  into  $C_3$  which are one-to-one and keep the neutral element 1 fixed, are the identity and the inversion. Both mappings are automorphisms of  $C_3$  and commute with each other.

**Lemma 1.** *All loops  $L = C_3 \rtimes (C_2)^n$  are power-associative.*

**Proof:** The restriction of  $\Phi$  and  $\Psi$  to a subloop which is generated by a single element is a homomorphism. Therefore  $L$  is power-associative by the preceding Remark.  $\square$

**Proposition 1.** *A semidirect product  $L = C_3 \rtimes (C_2)^n$  is a left respectively right Bol-loop if and only if  $\Phi$  respectively  $\Psi$  is a homomorphism.*

**Proof:** Because of Remark 1 the left Bol-identity yields:

$$\begin{aligned} & \left( a^{\Phi(qpr)} \left( b^{\Phi(pr)} \left( a^{\Phi(r)} c^{\Psi(p)} \right)^{\Psi(q)} \right)^{\Psi(p)} \right)_{pqpr} \\ &= \left( \left( a^{\Phi(qp)} \left( b^{\Phi(p)} a^{\Psi(q)} \right)^{\Psi(p)} \right)^{\Phi(r)} c^{\Psi(pqp)} \right)_{pqpr} \end{aligned} \quad (1)$$

If  $L$  is a left Bol-loop equation (1) implies for  $a = c = 1$  that  $\Phi$  is a homomorphism.

If  $\Phi$  is a homomorphism we obtain for the first component of (1):

$$a^{\Phi(qpr)} \left( b^{\Phi(pr)} \right)^{\Psi(p)} = \left( a^{\Phi(qp)} \right)^{\Phi(r)} \left( \left( b^{\Phi(p)} \right)^{\Phi(r)} \right)^{\Psi(p)} \quad (2)$$

which is valid for all  $a, b \in C_3$  and all  $p, q, r \in (C_2)^n$ . Therefore  $L$  is a left Bol-loop.

The proof for right Bol-loops is analogous.  $\square$

To classify the constructed loops up to isomorphism we now determine the order of the non-trivial elements in the loops.

**Lemma 2.** *Let  $L = C_3 \rtimes (C_2)^n$  be a loop,  $a \in C_3 \setminus \{1\}$  and  $p \in (C_2)^n \setminus \{1\}$ . Then the order of  $(a, p)$  is 2 if and only if  $\Phi(p) \neq \Psi(p)$  and 6 if and only if  $\Phi(p) = \Psi(p)$ .*

**Proof:** If  $\Phi(p) \neq \Psi(p)$  then  $(a, p)(a, p) = (1, 1)$  holds because of  $\Phi(p), \Psi(p) \in \{\text{id}, \text{inv}\}$ . If  $\Phi(p) = \Psi(p)$  then the first component of  $(a, p)^n$  is a power of  $a$  or  $a^{-1}$ . The second component alternates between 1 and  $p$ . Therefore the order of  $(a, p)$  is the least common multiple of 2 and 3.

Conversely if  $(a, p)(a, p) = (1, 1)$  then  $\Phi(p) \neq \Psi(p)$  because of  $\Phi(p), \Psi(p) \in \{\text{id}, \text{inv}\}$ . Assume the order of  $(a, p)$  to be 6 and  $\Phi(p) \neq \Psi(p)$ . This is a contradiction to the first part of the proof.  $\square$

**Proposition 2** ( $C_3 \rtimes (C_2)^2$ ). *Two proper loops of the form  $C_3 \rtimes (C_2)^2$  are isomorphic if and only if both loops have the same number of elements with order 6. A loop  $L = C_3 \rtimes (C_2)^2$  is a Bol-loop if and only if it has exactly zero or two elements of order 6.*

**Proof:** Lemma 2 implies that a loop is a Bol-loop if and only if it has exactly zero or two elements of order 6.

Let  $L_1$  and  $L_2$  be loops with the same number of elements with order 6. Then it can be shown that

$$\iota: \begin{cases} (a, p) \mapsto (a, p) & \text{if } \Phi_1(p) = \Phi_2(p) \\ (a, p) \mapsto (a^{-1}, p) & \text{if } \Phi_1(p) \neq \Phi_2(p) \end{cases}$$

is an isomorphism between  $L_1$  and  $L_2$ . The elements  $(a, p)$  with order 6 are assumed to have the same second component  $p \in (C_2)^2$  because loops can be transferred in this form by obvious (anti-)isomorphisms. By Lemma 2 this implies that  $\Phi_1(p) = \Psi_1(p)$  is equivalent to  $\Phi_2(p) = \Psi_2(p)$ .

Only the first component has to be analysed to check if  $\iota$  is an isomorphism. The validity of the equation

$$\iota_{pq}(a^{\Phi_1(q)} b^{\Psi_1(p)}) = (\iota_p(a))^{\Phi_2(q)} (\iota_q(b))^{\Psi_2(p)} \quad (3)$$

is shown by case analysis.

Since  $\Phi_1(p)$  in  $L_1$  can be different from  $\Phi_2(p)$  in  $L_2$  there are four cases. The mapping  $\Psi$  is not considered in the following because it is determined by the order of the elements and the choice of  $\Phi$ .

First the cases where  $\Phi_1(p)$  and  $\Phi_2(p)$  are unequal for all  $p$  or equal for exactly two elements  $p, q \in V_4$ : These loops can be trivially antiisomorphic by symmetry of  $\Phi$  and  $\Psi$ . Otherwise they are not (anti-)isomorphic because out of every other pair of loops, which satisfies the preconditions, one and only one loop is a Bol-loop. Therefore in this cases it is not necessary to prove the validity of equation (3).

If there is exactly one element  $r \in V_4$  for which  $\Phi_1(r) = \Phi_2(r)$ , then there are three possibilities, namely  $\Phi_1(pq) = \Phi_2(pq)$ ,  $\Phi_1(p) = \Phi_2(p)$  or  $\Phi_1(q) = \Phi_2(q)$ . In all three cases equation (3) holds for all combinations of  $\Phi_1(q)$ ,  $\Psi_1(p)$ ,  $\Phi_2(q)$ ,  $\Psi_2(p) \in \{\text{id}, \text{inv}\}$ .

In the last case, which is  $\Phi_1(p) = \Phi_2(p)$ ,  $\Phi_1(q) = \Phi_2(q)$  and  $\Phi_1(pq) = \Phi_2(pq)$ , the validity of equation (3) is obvious.  $\square$

**Corollary 1.** *There are 32 Bol-loops of the form  $C_3 \times (C_2)^2$  which are distributed in two classes of isomorphism.*

**Theorem 1** ( $C_3 \times (C_2)^n$ ). *Two proper Bol-loops of the form  $C_3 \times (C_2)^n$  are isomorphic if and only if they have the same number of elements with order 6.*

**Proof:** If  $L_1$  and  $L_2$  are proper Bol-loops of the form  $C_3 \times (C_2)^n$  then  $\Phi$  or  $\Psi$  is a homomorphism by Proposition 1. Without loss of generality we assume both loops to be left Bol-loops. If the loops have the same number of elements with order 6 then the mapping  $\iota$  as in the proof of Proposition 2 can be shown to be an isomorphism from  $L_1$  onto  $L_2$ : Any two elements  $\bar{a} = (a, p)$  and  $\bar{b} = (b, q)$  of  $C_3 \times (C_2)^n$  generate a subloop of  $C_3 \times (C_2)^n$  isomorphic to  $C_3 \times (C_2)^2$ . Therefore  $\iota$  is an isomorphism by the proof of Proposition 2.  $\square$

**Corollary 2.** *For  $n \geq 3$  the proper Bol-loops of the form  $C_3 \times (C_2)^n$  are distributed in  $2^n - 1$  classes of isomorphism.*

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